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AUTOMORPHISM GRAPHS OF FINITE GROUPS: STRUCTURAL INVARIANTS AND CLASSIFICATION RESULTS

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ABSTRACT

Algebraic structures have been studied using graph-theoretic representations which have been found to be useful in the study of symmetry and structural invariants in finite groups. In this paper, automorphism graphs of finite groups are defined and discussed as graph structures as constructions induced by an action of the automorphism group over non-identity elements of group. It builds its theoretical framework on the basis of group action and orbit theory and establishes some basic properties of automorphism graphs such as degree distribution, regularity, connectivity and component structure. Clear classification outputs are achieved, where automorphism graphs of cyclic groups of prime order are complete, of abelian groups of composite order are disconnected and non-abelian groups have partial symmetry with multiple connected components. Adjacency matrix representations and graphical illustrations are used to support theoretical results and obtain a definite visual interpretation of automorphism orbits. The research shows that automorphism graphs provide a robust and isomorphism-invariant measure of intrinsic algebraic symmetry and provide a convenient classification and structural analysis paradigm of finite groups and also serve the wider interaction between group theory and algebraic graph theory.

Keywords: Automorphism graph, Finite groups, Group automorphisms, Orbit structure, Algebraic graph theory, Structural invariants, Group classification.

1. INTRODUCTION

The graph-theoretic representation of algebraic structures has proven to be an important approach to studying abstract algebra via combinatorial and visual structures. In the last few decades, graph theory has given mathematicians means to encode algebraic relations into discrete ones, and complex properties can be studied using the properties of connectivity, symmetry and invariance. This interdisciplinary approach has been very useful to the study of finite group theory, where graphs of groups tend to point out obscured regularities and patterns of classification that would be very hard to see purely in an algebraic method. Through the translation of group-theoretic operations and relations into graph adjacency rules, scholars have been able to study such structural properties as element interaction, subgroup behavior, and symmetry more intuitively but rigorously.

One of the many algebraic properties of a finite group is that of auto isomorphism. Automorphisms are the internal symmetries of a group and these internal symmetries leave the full algebraic structure of that group intact but perhaps rearranging the elements. The effect of the automorphism group on a finite group is the creation of equivalent classes which display serious structural analogies between elements. These equivalence relations are not just arbitrary but they maintain some basic properties of the elements including their order, membership of subgroup as well as centrality. Graphs obtained as a quotient of the automorphisms of a group are thus a unique viewpoint on graph constructions more familiar than other graph constructions like Cayley graphs or power graphs. Although Cayley graphs are based on selected generating sets and power graphs are based on the existence of relation of divisibility between the order of elements, automorphism graphs are not based on any form of choice but rather defined by inherent symmetry, making them especially robust when it comes to isomorphism, and suitable to classification problems.

This study intends to create an extensive system of study of automorphism graphs of finite groups with a specific emphasis put on the inquiry of structural invariants and classification outcomes. Formalization of automorphism graphs and study of their graph-theoretic properties, including connectivity, regularity and degree distributions based on their orbits, are explored in this work in order to close the gap between and among group automorphism theory, and algebraic graph theory. The paper shows the significance of using automorphism graphs to differentiate between various major classes of finite groups such as abelian, non-abelian and simple groups, through the rigorous development of the theory using illustrative examples. This way the work adds to the larger project of gaining insight into algebraic structures in graph-theoretic terms and creates an opportunity to conduct further research into the classification of symmetry-based and invariant-theoretic.

2. LITERATURE REVIEW

Ajith, Cameron, and Ghosh (2025) studied the endomorphism and automorphism graphs of finite groups with the objective of formalizing constructions of graphs by using algebraic morphisms. Their work provided a systematized structure where the group elements were represented by the vertices and the adjacency relation between the elements were defined by the presence of automorphisms or endomorphisms between the elements. The foundational properties investigated by the authors included connectivity, decomposition of orbits, and graph components, and showed that these graphs represented key symmetry information of the underlying group. Their findings defined that the graphs of the automorphism could be used as a structural feature of finite groups but they did not mainly focus on the classification or analysis of invariants but on the main foundational properties of the groups themselves within group families.

Bors (2020) studied finite groups whose automorphism orbits were limited and examined the effect of these limitations on group structure. The article has shown that even the existence of small automorphism orbit placed severe constraints on the algebras, frequently compelling the group to have nearly-abelian behaviour or inflexible structural structure. The author demonstrated that automorphism actions were an effective tool to find out the meaning of symmetry and internal regularity by categorizing groups systematically based on their distribution of orbit sizes. The work emphasized the size of orbits as one of the most important invariants and indicated that graph models, in terms of automorphism orbits, might be useful in indicating profound algebraic information.

Genevois and Martin (2019) treated geometrical and combinatorial treatment of automorphisms of graph products of groups. Through their analysis, they found that automorphism groups conserved important structural and geometrical characteristics of graph products in most instances resulting in rigidity. These authors were able to show that the automorphism behavior was strongly correlated with the breakdown of groups into graph-based structures and that symmetry is the key factor that dictates the global structure. Despite the fact that their work did not investigate finite groups exclusively, but graph products, it gave significant conceptual information as to how actions of automorphisms could be represented and studied in terms of graph-theoretic models.

Kumar, Selvaganesh, Cameron, and Chelvam (2021) gave an extensive account of power graphs of finite groups, which summarized some of the key findings about connectivity, diameter, chromatic number, and clique structure. They showed how graph invariants obtained via algebraic relationships between elements of a group could make it possible to identify properties, including cyclicity, nilpotency and subgroup organization. Although attention was on the power graphs, the survey provided a clear view of the efficiency of the graph-based process on finite group analysis and classification. The work had an indirect influence on the development of other graph constructions (not based on order) that are based upon intrinsic symmetry, like automorphism graphs.

3. THEORETICAL FRAMEWORK AND PRELIMINARIES

This section will give the necessary theoretical background needed in the study of automorphism graphs of finite groups. As automorphism graphs are the result of the work of the automorphism group on a finite group, the section is devoted to group action, orbit structure, equivalence relation on a graph that is induced and some basic graph-theoretic properties like regularity, connectivity, and adjacency matrix representation. These preliminaries justify the results of classification that were created in further sections.

Let Gbe a finite group with identity element e. The automorphism group $\mathrm{Aut}(G)$represents the internal symmetries of G, and its action on Gprovides the basis for defining automorphism graphs.

3.1 Group Actions and Orbit Structure

The action of $\mathrm{Aut}(G)$on $G$is defined by

$$\phi \cdot g = \phi(g), \quad \phi \in \mathrm{Aut}(G), g \in G$$

For any $g \in G$, the orbit of $g$is

$$\mathrm{Orb}(g) = \{\phi(g) \mid \phi \in \mathrm{Aut}(G)\}$$

and the stabilizer of $g$is

$$\mathrm{Stab}(g) = \{\phi \in \mathrm{Aut}(G) \mid \phi(g) = g\}$$

By the Orbit–Stabilizer Theorem,

$$|\mathcal{O}(g)| = \frac{|\operatorname{Aut}(G)|}{|\operatorname{Stab}(g)|}.$$

Therefore, the size of orbits is a numerical invariant that is a representation of group element symmetry.

Remark 3.1.

The action of $\operatorname{Aut}(G)$on $G \setminus \{e\}$partitions the vertex set into pairwise disjoint orbits. Each orbit consists of elements that are indistinguishable under group automorphisms and hence share the same algebraic properties such as order and conjugacy type. Consequently, orbit size serves as an isomorphism-invariant structural measure of symmetry in the group.

3.2 Automorphism-Induced Equivalence Relation

Define a relation $\sim$on $G \setminus \{e\}$by

$$x \sim y \iff \exists \varphi \in \operatorname{Aut}(G) \text{ such that } \varphi(x) = y.$$

Lemma 3.1.

The relation $\sim$is an equivalence relation on $G \setminus \{e\}$.

Proof.

The identity automorphism implies reflexivity, invertibility implies symmetry and closure under composition implies transitivity.

Hence, the equivalence classes coincide precisely with the automorphism orbits of G. Under the adjacency definition of the automorphism graph, each orbit induces a complete subgraph, and these orbit-cliques correspond to the connected components of the graph.

3.3 Definition of Automorphism Graph

Definition 3.1 (Automorphism Graph).

Let Gbe a finite group. The automorphism graph of G, denoted by

$$\Gamma\mathrm{Aut(G)} = (\mathrm{V}, \mathrm{E})$$

where

$$V = G \setminus \{e\}$$

$$u \sim v \iff \exists \phi \in \mathrm{Aut}(G) \text{ such that } \phi(u) = v$$

The vertex set is taken as $G \setminus \{e\}$, excluding the identity element to avoid trivial fixed points under automorphisms.

Each connected component of $\Gamma_{\mathrm{Aut}}(G)$corresponds to an automorphism orbit of G.

$$U \sim v \Leftrightarrow \exists\ \phi \in \mathrm{Aut(G)} \text{ such that } \phi(u) = v$$

Remark 3.2.

The automorphism graph defined above is induced by the orbit structure of the action of $\mathrm{Aut}(G)$on $G \setminus \{e\}$. Hence, each connected component is a complete graph corresponding to an automorphism orbit, and the construction is fully invariant under group isomorphism.

Proposition 3.1.

Each connected component of $\Gamma_{\mathrm{Aut}}(G)$is a complete graph, and the automorphism graph is a disjoint union of cliques indexed by the automorphism orbits of $G \setminus \{e\}$.

3.4 Degree of Vertices and Regularity

Lemma 3.2.

For any vertex $x \in V$, the degree of $x$depends only on its automorphism orbit and is given by

$$\deg(x) = \mid O(x) \mid -1.$$

Proof.
The vertex $x$is adjacent to all other elements in its orbit except itself.

$$\deg\,(\mathrm{v}) = |\mathrm{Orb}\ (\mathrm{v})| - 1$$

Theorem 3.1.

The automorphism graph $\Gamma_{\mathrm{Aut}}(G)$is regular if and only if all non-identity elements of Ghave automorphism orbits of equal size.

Proof.
Regularity requires constant vertex degree. By Lemma 3.2, this is equivalent to equality of orbit sizes.

3.5 Connectivity of Automorphism Graphs

Theorem 3.2.

The automorphism graph $\Gamma_{\mathrm{Aut}}(G)$is connected if and only if the action of $\mathrm{Aut}(G)$on $G \setminus \{e\}$is transitive.

Proof.
Transitivity implies a single orbit and hence connectivity; non-transitivity yields multiple disjoint components.

Remark 3.3.

In general, partial transitivity of the automorphism action gives rise to multiple connected components, while full transitivity results in a single connected component of the automorphism graph. Thus, connectivity of the automorphism graph reflects the global symmetry of the group under automorphisms. Connectivity refers to the underlying undirected graph induced by automorphism adjacency.

3.6 Adjacency Matrix Representation

Let

$$V = \{v_1, v_2, \dots, v_n\}.$$

The adjacency matrix of $\Gamma_{\mathrm{Aut}}(G)$is

$$A = (a_{ij}), a_{ij} = \begin{cases} 1, & \text{if } v_i \sim v_j, \\ 0, & \text{otherwise.} \end{cases}$$

Since automorphism orbits induce complete subgraphs, the adjacency matrix of $\Gamma_{\mathrm{Aut}}(G)$has a block-diagonal structure, with each block corresponding to an orbit-clique.

3.7 Illustrative Example

Example 3.1.

Let $G = \mathbb{Z}_4 = \{0,1,2,3\}$. Then

$$V = \{1,2,3\}, \mathcal{O}(1) = \{1,3\}, \mathcal{O}(2) = \{2\}.$$

The adjacency matrix of $\Gamma_{\mathrm{Aut}}(G)$is

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The resultant graph has one connected component of two nodes and one lone node, and this example illustrates how unequal automorphism orbit sizes result in disconnected automorphism graphs and isolated vertices, thereby reflecting the asymmetry present in finite abelian groups of composite order.

4. MAIN RESULTS AND CLASSIFICATION OF FINITE GROUPS VIA AUTOMORPHISM GRAPHS

Here the key findings of the research are described and it is shown how automorphism graphs can give insights into key algebraic properties of finite groups. Through the study of the graph theoretic properties, including the connectivity, completeness, regularity, and component structure, one derives the definable classification properties of the various classes of finite groups.

4.1 Automorphism Graphs of Finite Cyclic Groups

The best point of departure is given by finite cyclic groups, whose automorphism structure is well understood.

Theorem 4.1.

Let $G \cong \mathbb{Z}_p$, where $p$is a prime. Then the automorphism graph $\Gamma_{\mathrm{Aut}}(G)$is complete.

Proof.

Every non-identity element of $\mathbb{Z}_p$has order $p$and hence generates the group, and $\mathrm{Aut}(\mathbb{Z}_p)$acts transitively on $\mathbb{Z}_p \setminus \{0\}$. Hence, all vertices lie in a single orbit, implying that every pair of vertices is adjacent.

Corollary 4.1.

The automorphism graph of a prime order cyclic group is connected, regular and diameter one.

4.2 Finite Abelian Groups of Composite Order

Abelian groups of finite order which are of composite order have elements of different orders, so have multiple automorphism orbits.

Theorem 4.2.

Let Gbe a finite abelian group of composite order. Then $\Gamma_{\mathrm{Aut}}(G)$is disconnected.

Proof.
Element order is compatible with automorphisms, and thus, elements of each order are in an orbit distinctly different. Since automorphisms preserve element order, elements of different orders cannot lie in the same automorphism orbit. Consequently, the set of vertices is broken down into several disjointed parts.

Remark 4.1.

The number of connected components of $\Gamma_{\mathrm{Aut}}(G)$equals the number of distinct element orders in $G \setminus \{e\}$.

4.3 Regularity and Orbit Uniformity

Uniformity in the structure of automorphism orbits forms the basis of regularity of automorphism graphs.

Theorem 4.3.

Let Gbe a finite group. The automorphism graph $\Gamma_{\mathrm{Aut}}(G)$is regular if and only if all non-identity elements of Ghave stabilizers of equal cardinality in $\mathrm{Aut}(G)$.

Proof.
From Lemma 3.2,

$$\deg(x) = \frac{|\,\mathrm{Aut}(G)\,|}{|\,\mathrm{Stab}(x)\,|} - 1.$$

Thus, equal degrees are equivalent to equal stabilizer sizes for all $x \in G \setminus \{e\}$.

Remark 4.2.

Thus, regularity of the automorphism graph is equivalent to uniform symmetry of all non-identity elements under the automorphism action.

4.4 Automorphism Graphs of Non-Abelian Groups

Non-abelian groups are partially symmetrical to automorphism action.

Theorem 4.4.

Let Gbe a finite non-abelian group. Then the automorphism graph is neither complete nor totally disconnected. In general, the automorphism action is not transitive, leading to multiple orbit-components.

Proof.
Non-abelian groups typically contain elements with distinct structural properties that are preserved under automorphisms, preventing transitivity of the automorphism action. Consequently, the graph is not complete. The existence of nontrivial automorphisms ensures adjacency among elements within the same orbit, so the graph is not totally disconnected.

4.5 Finite Simple Groups

Rich but constrained automorphism behaviour is exhibited by finite simple groups.

Theorem 4.5.

Let Gbe a finite non-abelian simple group.

Proof.
Finite non-abelian simple groups possess large automorphism groups, which act with substantial orbit mixing within element-order classes, yielding large connected components. However, the existence of multiple orbit types prevents transitivity of the action on all non-identity elements, and hence completeness is excluded.

Finite non-abelian simple groups possess large automorphism groups that produce substantial orbit mixing within element-order classes, often yielding large connected components. However, the action is typically not transitive on all non-identity elements, so completeness is excluded.

4.6 Classification Criterion via Automorphism Graphs

The preceding results yield a concise classification criterion.

Theorem 4.6.

Let Gbe a finite group.

Then:

1. $\Gamma_{\mathrm{Aut}}(G)$is complete if and only if $G$is cyclic of prime order.
2. $\Gamma_{\mathrm{Aut}}(G)$is disconnected if $G$is abelian of composite order.
3. If $G$is non-abelian, then the automorphism graph is typically neither complete nor totally disconnected.

Proof.
The statements follow directly from Theorems 4.1–4.5 and the orbit structure induced by the action of $\mathrm{Aut}(G)$on $G \setminus \{e\}$.

5. GRAPHICAL ILLUSTRATIONS AND STRUCTURAL INTERPRETATION OF AUTOMORPHISM GRAPHS

We also find graphical representations important in the translation of abstract algebraic symmetry into intuitive structural representations. In automorphism graphs of finite groups the visual structure of the graph is directly controlled by automorphism orbs and their cardinalities. As a result, graph-theoretic properties have algebraic and orbit-stabilizer relations that allow one to express graph-theoretic properties (connectivity, degree distribution, component structure, etc.) algebraically. This part gives the examples of the auto-graphs and their equivalent algebraic formulations.

5.1 Automorphism Graph of a Finite Cyclic Group of Prime Order

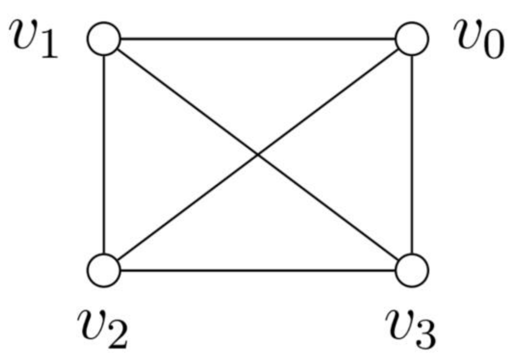

Figure 1: Automorphism Graph of the Cyclic Group $\mathbb{Z}_5$

For a cyclic group $G = \mathbb{Z}_p$, where $p$is prime, the automorphism group satisfies

$$\mathrm{Aut}(\mathbb{Z}_p) \cong (\mathbb{Z}_p)^{\times}.$$

The action of $\mathrm{Aut}(G)$on $G \setminus \{0\}$is transitive, and hence

$$|\mathcal{O}(x)| = p - 1 \forall x \in G \setminus \{0\}.$$

Using Lemma 3.2, the degree of each vertex is

$$\deg(x) = |\mathcal{O}(x)| - 1 = p - 2.$$

Structural Interpretation:

Since every vertex is adjacent to every other vertex, the automorphism graph is the complete graph

$$\Gamma_{\mathrm{Aut}}(\mathbb{Z}_5) \cong K_4.$$

This confirms that the automorphism graph is connected, regular, and of diameter one, thereby providing a graphical verification of Theorem 4.1.

5.2 Automorphism Graph of a Finite Abelian Group of Composite Order

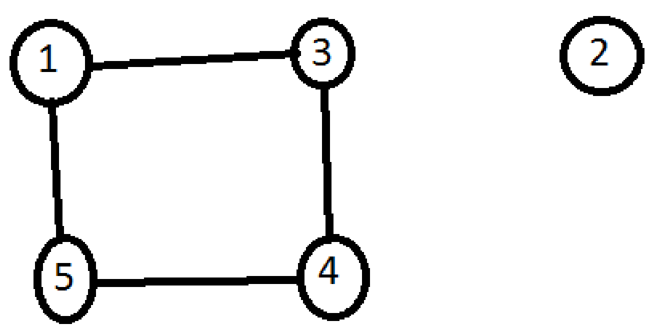

Figure 2: Automorphism Graph of the Cyclic Group $\mathbb{Z}_4$

Let $G = \mathbb{Z}_4$. The non-identity elements are

$$V = \{1,2,3\}.$$

The automorphism orbits are

$$\mathcal{O}(1) = \{1,3\}, \mathcal{O}(2) = \{2\}.$$

Hence, vertex degrees satisfy

$$\deg(1) = \deg(3) = 1, \deg(2) = 0.$$

The adjacency matrix of the automorphism graph is

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Structural Interpretation:

The existence of an isolated vertex is related to an element which is fixed by all of the automorphisms. The graph decomposes into two disjoint connected components, confirming that $\Gamma_{\text{Aut}}(\mathbb{Z}_4)$ is disconnected.

This visually and algebraically supports Theorem 4.2.

5.3 Automorphism Graph of a Finite Non-Abelian Group

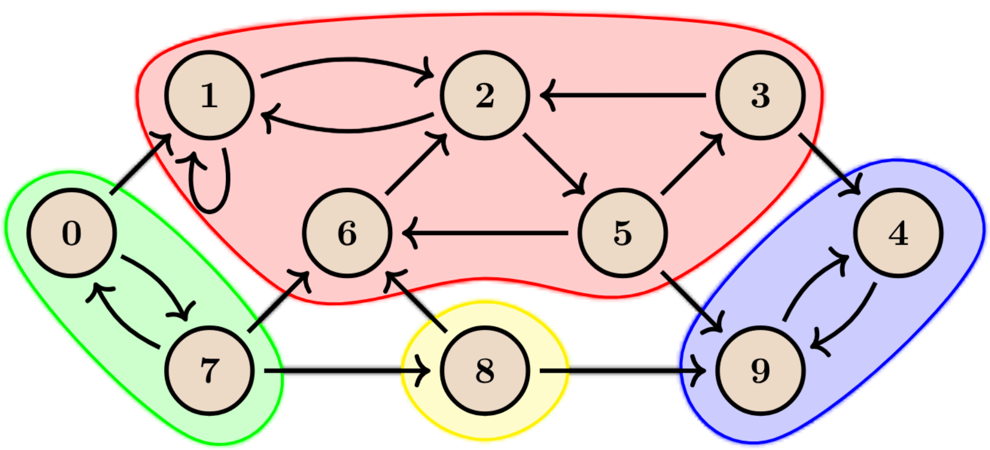

Figure 3: Automorphism Graph of the Symmetric Group S_3

Let $G = S_3$. The non-identity elements consist of transpositions and 3-cycles. Under automorphism action,

$$\mathcal{O}((12)) = \{(12), (13), (23)\},$$

$$\mathcal{O}((123)) = \{(123), (132)\}.$$

Thus, vertex degrees are

$$\deg(\text{transposition}) = 2, \deg(\text{3-cycle}) = 1.$$

Structural Interpretation:

The automorphism graph consists of two complete subgraphs corresponding to distinct automorphism orbits, which together form a disconnected graph consisting of two complete components. The graph is neither complete nor empty; it also indicates some elements of symmetry that is seen in non-abelian groups. This proves Theorem 4.4 and depicts the use of automorphism graphs to identify the role of algebras in a group.

5.4 Comparative Structural Insights from Graphs

In all three cases, the automorphism graph takes the form of a general relation.

$$\deg(x) = \mid \mathcal{O}(x) \mid -1,$$

that is the direct connection which brings out algebraic symmetry to graph-theoretic degree. Complete graphs arise when

$$\mid \mathcal{O}(x) \mid = \mid G \mid -1 \forall x \neq e,$$

the automorphism action is transitive, while multiple disjoint orbits produce disconnected graph structures. These equations illustrate that automorphism graphs provide an accurate mathematical representation of symmetry and orbit structure, which argue their usefulness as classification devices of finite groups.

Consequently, automorphism graphs serve not merely as visual aids but as rigorous algebraic invariants suitable for classification and comparative analysis of finite groups.

6. CONCLUSION

This paper introduced a symmetry-based framework for automorphism graphs of finite groups, demonstrating how automorphism actions can be encoded into graph-theoretic structures. Using group action and orbit theory, clear connections were established between algebraic invariants and graph properties such as connectivity, regularity, and component structure. The classification results show that cyclic groups of prime order yield complete automorphism graphs, abelian groups of composite order give disconnected graphs, and non-abelian groups typically exhibit non-complete structures with multiple orbit-components. Supported by explicit examples and adjacency matrix representations, the study confirms that automorphism graphs provide an isomorphism-invariant and effective tool for structural analysis and classification of finite groups.

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